What’s math got to do with patterns on fish?

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Abstract

When you think of fish, what comes to mind? Do you think of math? Probably not. Maybe you think of pet goldfish, animated characters like Dory or Nemo, or trout in a local river. One of the things that all these fish have in common is patterns on their skin. Nemo sports black and white stripes on his orange skin, and trout have spots. Even goldfish have a pattern – it’s just plain gold (and kinda boring). Why do some fish have stripes, others have spots, and others have plain patterns? It turns out that this is a tricky question, so scientists use tools from several subjects to answer it. In this paper, we combine biology with math and computer coding to help figure out how fish get different skin patterns.

What are fish patterns made of?

Many types of animals have skin patterns. Zebras have stripes and leopards have spots, but fish display all sorts of different patterns. When you look at Figure 1, what patterns do you see? There are fish with stripes, spots, mazes, and plain patterns [5]. Which pattern is your favorite? Scientists are working to figure out why fish have different patterns. Zebrafish (Figure 1d) are good for doing experiments with, so a lot of biologists like zebrafish best [2, 9]. As you might guess from their name, zebrafish usually have dark and light stripes.

When we look at zebrafish patterns from far away, we see black and yellow stripes. If we put a zebrafish under a microscope, however, we can see that their patterns are made up of tiny dots, or [pigment cell]. Some of these cells are black (biologists call them melanophores) and others are yellow (their fancy name is xanthophores). There are other colors of pigment cells out there, too, and even humans have [pigment cell] in their skin.
Figure 1: Fish have all sorts of different skin patterns, and here are some examples. Their patterns are made up of pigment cells. (Images from [2, 5, 8, 9]; see end of document for sources.)

Cells follow rules to create stripes

We know that pigment cells make up fish patterns. But what’s really cool is that these cells have the job of creating stripes as zebrafish grow [7]. In this movie [7] of a fish growing, do you see the black cells moving around? There are yellow cells there too, but they are harder to see. You can think of each pigment cell in zebrafish skin as a person in a group. You can move around in the group, and cells can move around, too.

Black and yellow cells move and act in specific ways (or follow rules) to create patterns. Yellow cells following black cells is one of these rules. It’s almost like they are playing tag, you’re it! You can see a

1Watch a fish pattern form here: https://www.pnas.org/content/104/12/4790/tab-figures-data
**Thinking about fish patterns with biology and math**

**Cells in fish skin**

![Yellow cell](image1)

![Black cell](image2)

**Cells in a simulation**

**Mathematical model**

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
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<tbody>
<tr>
<td>3</td>
<td>4</td>
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</table>

**Our rule:** 1 side different, the rest the same

For each cell, color 1 side different from it and the rest the same color as it.

**Examples:**

- These cells follow our rule.
- These cells do not.

**Figure 2:** Cells follow rules to create patterns. To simulate patterns with a mathematical model on a computer, think of fish skin as a checkerboard [3]. Then pick a rule for what color of cell goes in each square. Our rule is “One side different, the rest the same.” If you put your finger on a cell in the center, one of its neighbors should have a different color than it, and the rest should be the same color as it.

black cell leading the way and a yellow cell following it in this movie [4]. Crazy!

**How do scientists study fish patterns?**

While some of the rules for how cells behave in zebrafish are known (like yellow cells following black cells [4]), many of these rules are a mystery and still need to be discovered. Cells are tiny, and it’s tough to see them and catch their interactions on the fish skin. Zebrafish also take a long time to grow their patterns (a couple months) [2], so biologists have to be patient with their experiments.

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Watch yellow cells following black cells here: [https://www.youtube.com/watch?v=0wECUnwgN8A](https://www.youtube.com/watch?v=0wECUnwgN8A)
<table>
<thead>
<tr>
<th>Problem number</th>
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<tr>
<td>1</td>
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<td>9</td>
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</tbody>
</table>

Table 1: Determine the option that follows the [mathematical model] in Figure 2. Solutions: 1b, 2b, 3a, 4c (one side has to be yellow, but it could be the right or bottom side), 5a (this cell is at the edge of the grid and only has 3 sides, so we have enough information this time), 6b, 7b, 8b, 9b.

This is where math and computer coding enter the picture. Math is all about patterns, rules, and puzzles, and it can be combined with biology to help figure out how cells behave [1, 3]. Mathematicians and biologists use different tools to study the same problems. Bringing their different ideas together makes the mystery of how zebrafish get their patterns less difficult to solve.

While biologists study zebrafish by looking into microscopes and doing experiments, mathematicians write down rules for how cells might behave [1, 3]. These rules and equations are called a [mathematical model] — they are a way of describing something in a mathematical way. It is important to remember that models are a good guess, but they could be wrong. To test their rules, mathematicians use computer coding to [simulate] fish patterns. This is like growing fish on a computer! One good thing about this is that simulating fish patterns only takes a few minutes, not a few months. (But we wouldn’t recommend trying to eat a simulated fish for dinner :-)

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Building a model of zebrafish stripes

Now we are ready to build a mathematical model of zebrafish patterns! Our first step is to describe fish skin as a checkerboard (see Figure 2). Each square in our checkerboard can either be empty (white), have a yellow cell, or contain a black cell.

To simulate fish patterns, we need to tell the black and yellow cells where to appear in the checkerboard. In other words, we need to describe their rules. Biologists know that black cells mostly like to be near other black cells, and yellow cells like to have mostly yellow cells around them. We use this knowledge to build our mathematical model.

In Figure 2, each cell in the checkerboard has 4 sides (unless the cell is at a corner or an edge). Our rule is “One side different, the rest the same.” This means that if you put your finger on a yellow cell, one of the cells that it touches should be black (different) and the other cells should be yellow (the same). You can see some examples of cells that follow this rule and cells that don’t in Figure 2. To get some practice with this rule, try filling in the worksheet in Table 1.

Our puzzle: Simulating zebrafish stripes

What patterns form when we use the rule “One side different, the rest the same”? You can try this out and build a fish pattern using Figure 3. If you print Figure 3 and cut out the black and yellow cells, then follow these steps:

- **Step 1:** Zebrafish have one yellow stripe when they are babies, so start by putting 2 strips of yellow cells at the top of your fish checkerboard.

- **Step 2:** Fill out the rest of the checkerboard by following the rule “One side different, the rest the same.”

Start by filling out the squares at the top of the checkerboard and slowly work down the fish from left to right. What pattern forms?

By following the steps in Figure 3, you are doing the same steps by hand that a computer would do to simulate a fish pattern. We’ve used computer coding to create fish patterns here—check out this website to simulate a fish yourself!

Why is studying fish patterns important, anyway?

Fish patterns might be nice to look at, but why do biologists and mathematicians study them? That’s a really good question, and the answer has to do with mutant patterns, medicine, and humans.

Turns out that zebrafish only usually have stripes (hence the “zebra” part of their name). As you know, stripes form because cells follow specific rules. But what happens if cells don’t follow their normal rules? This can happen if a fish has a mutation in its genes [6, 9]. In Figure 1, image (h) is not the only picture of a zebrafish. Believe it or not, images (d) and (e) are also zebrafish, but they are mutants!
Figure 3: Use this image to create a fish pattern by combining biology and math. Fill the checkerboard on the fish by following Steps 1 and 2. After you finish, print this page again. You can be an applied mathematician yourself – make up your own rules to discover other patterns that you can create! How would you change Step 2 to make a goldfish? You can also see how math is combined with computer coding to simulate different patterns at this [website](#).
Mutant zebrafish have different patterns because their cells follow different rules. A fish’s genes specify these rules, so mutant genes lead to mutant rules and mutant patterns. Humans can also have mutations in their genes and this can lead to disease.

Zebrafish and humans look very different, but they actually have a lot of similar genes. And that’s why studying zebrafish patterns is important. If we can figure out how cells behave to create normal and mutant patterns in fish, we might be able to better understand normal and unhealthy behavior of cells in humans down the line too.

**Now you get to be the applied mathematician!**

Scientists are working hard to discover the ways that black and yellow cells behave in normal zebrafish, and to figure out how these cells change their behavior in mutant fish. For mathematicians who study zebrafish, their job is to think of a fish pattern (like stripes) and then find the rules that can create that pattern. This is a puzzle.

How would you change the rule “One side different, the rest the same” to create a plain yellow fish? What rules and starting points create diagonal stripes? Try out your own rules by printing Figure 3 out again. By working on these questions, you are taking on the role of mathematician yourself! What fish patterns can you create with math?

**Glossary**

**genes** The biological instructions that an animal gets from their parents to specify their features (like stripe or spot patterns for a fish, or green or brown eyes for a person).

**mathematical model** Rules or equations that describes something in a mathematical way. In this article, the rule “One side different, the rest the same” is part of our mathematical model.

**mutation** A change in an animal’s genes from the normal setting to something different. For example, zebrafish usually have stripes (Figure 1h), but mutant zebrafish (Figures 1d, 1e) have spot or maze patterns.

**pigment cell** Cells are small units that all living things are made of (we are made of cells, and so is fish skin). Pigment cells are special cells that contain pigment (color).

**simulate** To imitate a real-world system on a computer. You can simulate fish patterns on this website with computer coding and a mathematical model, which provides directions to the computer.

**References**


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Author Biographies

Blake Shirman is a new Master’s student at DePaul University, and he hopes to one day get a Ph.D. in pure math. He spends his time looking at logic puzzles and games to see if math can make things easier, sometimes in fun and unexpected ways! One of his favorite puzzles is trying to predict cell patterns (like how to place checkers on a checkerboard with special rules about where they go). It wasn’t until he met Alexandria that Blake learned diagrams like this were being used to describe the vibrant stripes and spots on many of our undersea friends!

Alexandria Volkening got her Ph.D. in applied mathematics from Brown University, and now she’s an NSF–Simons Fellow at Northwestern University. She uses math to study all sorts of stuff (fish patterns, of course, but also elections, social media, and crowds — lots of things follow rules of behavior that can be described with math!). Her favorite fish to simulate is zebrafish, but her favorite fish for dinner is salmon. Alexandria is excited about studying more animal patterns, and she hopes that one day she can work with biologists to find out how clownfish (like Nemo) get their stripes!